TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 146, December 1969

THE COHOMOLOGY OF THE COMPLEX PROJECTIVE STIEFEL MANIFOLD

BY
CARLOS ALFREDO RUIZ[1]

0. Let U_n be the unitary group of order n. We have inclusions

$$\cdots \supset U_n \supset U_{n-1} \supset \cdots U_1 \approx S^1.$$

We denote by $W_{n,n-m}$ the complex Stiefel manifold $W_{n,n-m}=U_n/U_m$ and by $Y'_{n,n-m}$ the complex projective Stiefel manifold which we defined as follows: S^1, regarded as the set of complex numbers of module 1, acts by multiplication on U_n. This action being compatible with the above inclusions defines an action of S^1 on $W_{n,n-m}$ and we define $Y'_{n,n-m}$ as the set of orbits.

In particular we have

$$\begin{aligned} W_{n,1} &= U_n/U_{n-1} \approx S^{2n-1}, \\ W_{n,n} &= U_n, \\ Y'_{n,n} &= PU_n, \quad \text{the projective unitary group.} \end{aligned}$$

In this paper we compute $(H^* Y'_{n,k})$. Baum and Browder [1] have obtained our result in the special case $n=p^r$, $m=0$.

In order to state our main result we need some notation: Let ω be the generator of $H^*(CP^\infty)=\mathbf{Z}[\omega]$ and z_i the generators of $H^*(W_{n,k})=\bigwedge(z_{m+1},\ldots,z_n)$. Let $b_i=\text{G.C.D.}((C_{n,m+1},\ldots,C_{n,i}))$. Finally in §1, we will show there is a fibration

$$\text{(B)} \qquad W_{n,n-m} \xrightarrow[i]{} Y_{n,n-m} \xrightarrow[\pi]{} CP^\infty$$

with $Y_{n,n-m}$ of the same homotopy type as $Y'_{n,n-m}$. Then our main theorem is

Theorem A.

$$H^*(Y_{n,n-m}) = \mathbf{Z}[y]/I \otimes \bigwedge(v_{m+2},\ldots,v_n), \quad \textit{where } \pi^*\omega = y;$$

$i^*v_i=(b_{i-1}/b_i)z_i$ and I is the ideal generated by $b_i y^i$, $i=m+1,\ldots,n$.

In §1 we compute $H^*(Y_{n,k};\mathbf{Z}_p)$ and $H^*(Y_{n,k};\mathbf{Q})$ following the Gitler and Handel proof for real case [3]. In §2 we determine Ker π^* and Im i^*. In §3 we show that this information is enough to determine all relevant Bockstein homomorphisms

Received by the editors June 13, 1969.

[1] Supported by Consejo Nacional de Investigaciones Cientificas y Tecnicas de la Republica Argentina.

and compute the Bockstein spectral sequence for every prime p. This determines completely $H^*(Y_{n,k})$.

We wish to thank Professors M. Mahowald and S. Gitler for their helpful suggestions and comments.

1. First we construct a space $Y_{n,k}$ of the same homotopy type as $Y'_{n,k}$.

Observe that we have a principal bundle $S^1 \to W_{n,n-m} \to Y'_{n,n-m}$.

Let ξ be the associate line bundle.

PROPOSITION 1. *$n\xi = \xi \oplus \xi \oplus \cdots \oplus \xi$ has $n-m$ $\mathbf{C}$-linearly independent sections and it is the universal bundle for bundles $n\zeta$ having $n-m$ $\mathbf{C}$-linearly independent sections, where ζ is a line bundle.*

Proof. The proof is similar to the real case given in [3].

The inclusion $U_m \subset U_n$ gives rise to a fibration of classifying spaces

(A) $$W_{n,n-m} \xrightarrow[i_1]{} BU_m \xrightarrow[\pi_1]{} BU_n$$

and the transgression satisfies

(1) $$\tau z_i = \sigma_i \quad \text{where } \sigma_i \text{ is the universal Chern class.}$$

Now let γ be the canonical line bundle over CP^∞ and let f_n be the classifying map of $n\gamma$ and let

(B) $$W_{n,n-m} \xrightarrow[i]{} Y_{n,n-m} \xrightarrow[\pi]{} CP^\infty$$

be the fibration induced from (A) by f_n.

It is easy to see that the bundle induced by $f_n\pi$ is a universal bundle for the n-plane bundles satisfying the conditions of Proposition 1. Thus we have

PROPOSITION 2. *$Y_{n,n-m}$ and $Y'_{n,n-m}$ have the same homotopy type.*

From (1) and naturality of Chern classes and transgression

(2) $$\tau z_i = C_{n,i}\omega^i.$$

If $x \in H^*(E)$ we denote by $\bar{x}$ (resp. $\tilde{x}$) its image in $H^*(E; \mathbf{Z}_p)$ (resp. $H^*(E; \mathbf{Q})$).

Let $N(p)$ be the smallest i such that $m+1 \leqq i < n$ and $C_{n,i}$ is not zero, mod p. The following theorem is similar to [3, Theorem 1.6].

THEOREM 3.

$$H^*(Y_{n,n-m}; \mathbf{Z}_p) = \mathbf{Z}_p[\bar{y}]/[y^{N(p)}] \otimes \bigwedge (\bar{x}_{m+1} \cdots \widehat{\bar{x}_{N(p)}} \cdots \bar{x}_n),$$
$$H^*(Y_{n,n-m}; \mathbf{Q}) = \mathbf{Q}[y]/[y^{m+1}] \otimes \bigwedge (\tilde{x}_{m+2} \cdots \tilde{x}_n),$$

where p is a prime and $\mathbf{Q}$ is the set of rational numbers; $i^*(\bar{x}_i) = \bar{z}_i$; $\pi^*(\bar{\omega}) = \bar{y}$; and $i^*(\tilde{x}_i) = \tilde{z}_i$, $\pi^*(\tilde{\omega}) = \tilde{y}$.

Proof. Let K be a field, $K=\mathbf{Z}_p$ or $K=\mathbf{Q}$ and let N be the smallest i such that $m+1 \leqq i < n$ and $C_{n,i}$ is not zero in K.

In the spectral sequence of (B) with coefficients K we have

$$E_2 = K[\omega] \otimes \bigwedge (z_{m+1} \cdots z_n).$$

By (2) $d_r z_i = 0$, $r < 2i$, $d_{2i} z_i = \tau z_i = C_{n,i}\omega^i$, then $d_r = 0$, $r < 2N$ and then $E_2 = E_{2N}$; but

$$d_{2N} z_N = C_{n,N}\omega^N \neq 0,$$

thus, z_N does not survive in E_{2N+1} and the image of the ideal $[\omega^N]$ in E_{2N+1} is 0. Moreover, E_{2N+1} is still a tensor product:

$$E_{2N+1}^{*,0} \otimes E_{2N+1}^{0,*} = E_{2N+1}^{*,*}.$$

Now the following transgressions are zero (thus all the following differentials are zero), hence

$$E_\infty = E_{2N+1} = K[\omega]/[\omega^N] \otimes \bigwedge (z_{m+1} \cdots \hat{z}_N \cdots z_n).$$

The theorem now follows from Borel [2].

COROLLARY 4. $H^*(Y_{n,n-m})$ has p-torsion if and only if p divides $C_{n,m+1}$.

2. In this section we obtain the first results about $H^*(Y_{n,k})$. The key is Proposition 5 below on Ker π^*. In Corollary 7 we pick some elements in $H^*(Y_{n,k})$ and using them we choose new generators for $H^*(Y_{n,k}; \mathbf{Z}_p)$ and $H^*(Y_{n,k}; \mathbf{Q})$ ((7), (7′)).

PROPOSITION 5.

$$\operatorname{Ker} \pi^* = [b_{m+1}\omega^{m+1}; \ldots, b_n\omega^n].$$

COROLLARY 6.

$$\operatorname{Ker} \bar{\pi}^* = [b_{m+1}\omega^{m+2}; \ldots, b_n\omega^{n+1}].$$

Let $c_i = b_{i-1}/b_i$, $i = m+2, \ldots, n$.

COROLLARY 7.

$$T^{2m+1}/\operatorname{Im} i^* = \mathbf{Z}, \qquad T^q/\operatorname{Im} i^* = 0, \qquad q \geqq 2n,$$
$$T^{2i-1}/\operatorname{Im} i^* = \mathbf{Z}_{c_i}, \qquad m+2 \leqq i \leqq n.$$

Thus, there exist elements v_i in $H^*(Y_{n,n-m})$ such that $i^* v_i = c_i z_i$, $m+2 \leqq i \leqq n$.

Before we give the proof we recall some facts about transgression in the spectral sequence of a fibration.

We use the transgression in (B) as defined in the following diagram, $\tau = j^* \circ (\bar{\pi}^*)^{-1} \circ \delta$

$$\begin{array}{ccccccc}
H^*(Y_{n,n-m}) & \xrightarrow[i^*]{} & H^*(W_{n,n-m}) & \xrightarrow[\delta]{} & H^*(Y_{n,n-m}, W_{n,n-m}) & \longrightarrow & H^*(Y_{n,n-m}) \\
 & & & & \uparrow \bar{\pi}^* & & \uparrow \pi^* \\
 & & & & H^*(CP^\infty, *) & \xrightarrow[j^*]{} & H^*(CP^\infty).
\end{array}$$

We recall that in the spectral sequence of (B) $E_2^{0,2i-1} \approx H^{2i-1}(W_{n,n-m})$. This isomorphism carries the subgroup $E_{2i}^{0,2i-1}$ onto T^{2i-1}, the subgroup of transgressive elements. Also $E_2^{2i,0} \approx H^{2i}(CP^\infty)$ and this isomorphism induces the isomorphism of quotient groups $E_{2i}^{2i,0} \approx H^{2i}(CP^\infty)/\mathrm{Ker}^{2i}\,\bar{\pi}^*$. Moreover, via these isomorphisms, τ corresponds to $d_{2i}^{0,2i-1}$ and

$$\text{(3)} \qquad \mathrm{Im}\, d_{2i}^{0,2i-1} \approx \mathrm{Ker}^{2i}\,\pi^*/\mathrm{Ker}^{2i}\,\bar{\pi}^* \subset H^{2i}(CP^\infty)/\mathrm{Ker}^{2i}\,\bar{\pi}^*.$$

Finally, τ induces an isomorphism

$$\text{(4)} \qquad T^{q-1}/\mathrm{Im}\, i^* \approx \mathrm{Ker}^q\,\pi^*/\mathrm{Ker}^q\,\bar{\pi}^*.$$

Consider the diagram:

$$\text{(C)} \qquad \begin{array}{ccccc} H^*(W_{n,n-m}) & \xleftarrow{i^*} & H^*(Y_{n,n-m}) & \xleftarrow{\pi^*} & H^*(CP^\infty) \\ \downarrow \theta & & \downarrow \theta & & \downarrow \theta \\ H^*(W_{n,n-m}; A) & \xleftarrow{i_A^*} & H^*(Y_{n,n-m}; A) & \xleftarrow{\pi_A^*} & H^*(CP^\infty; A). \end{array}$$

If $A = \mathbf{Z}_p$, then Theorem 2 and $i^* \circ \pi^* = 0$ yield

$$\text{(5)} \qquad \mathrm{Ker}\, i^*_{\mathbf{Z}_p} = [\bar{y}],$$

$$\text{(6)} \qquad \mathrm{Ker}\, \pi^*_{\mathbf{Z}_p} = [\omega^{N(p)}].$$

If $A = \mathbf{Q}$, we have

$$\text{(5')} \qquad \mathrm{Ker}\, i^*_{\mathbf{Q}} = [\bar{y}],$$

$$\text{(6')} \qquad \mathrm{Ker}\, \pi^*_{\mathbf{Q}} = [\bar{\omega}^{m+1}].$$

Proof of Proposition 5. The spectral sequence of (B) is trivial through E_{2m+2}. Thus $E_{2m+2}^{2m+2,0} \approx H^{2m+2}(CP^\infty)$ and $\mathrm{Ker}^q\,\bar{\pi}^* = 0$, $q \leqq 2m+2$.

From (2) and (3) we obtain

$$\mathrm{Ker}^q\,\pi^* = 0, \qquad q < 2m+2 \quad \text{and} \quad \mathrm{Ker}^{2m+2}\,\pi^* = [C_{n,m+1}\omega^{m+1}]^{2m+2}.$$

Applying (2) repeatedly we have

$$[b_{m+1}\omega^{m+1}, \ldots, b_n\omega^n] \subset \mathrm{Ker}\,\pi^* \subset H^*(CP^\infty).$$

For the other inclusion, put $b_i = ap^r$, where p does not divide a. We use diagram (C) with $A = \mathbf{Z}_{p^r}$, if $p^s c\omega^i$ belongs to Ker π^*, $s < r$ and c divides a, then $p^s c\theta(\omega)^i$ belongs to Ker $\pi^*_{\mathbf{Z}_{p^r}}$ but it is not 0 because c is not a divisor of 0 in $\mathbf{Z}_{p^r}$. On the other hand in the spectral sequence of (B) with coefficients $\mathbf{Z}_{p^r}$

$$\tau z_k = C_{n,k}\omega^k = 0, \qquad k < i,$$

because p^r divides $C_{n,k}$ for those k. Thus, the spectral sequence is trivial through $2i$ and then $\mathrm{Ker}^{2i}\,\pi^*_{\mathbf{Z}_{p^r}} = 0$.

Proof of Corollary 6. Follows from (3) and Proposition 5.

Proof of Corollary 7. First part follows from (4), second part follows trivially from first part.

The elements v_i are not unique, we will choose a fixed set of such elements arbitrarily.

In diagram (C) with $\mathbf{Z}_p = A$, we have:

(i) if p does not divide c_i, $\theta v_i = c_i\bar{x}_i + u_i$, $m+2 \leqq i \leqq n$ where $u_i \in \operatorname{Ker} i^*_{\mathbf{Z}_p}$;

(ii) if p does divide c_i, then $\theta v_i \in \operatorname{Ker} i^*_{\mathbf{Z}_p}$.

Let I be $I = \{i;\ p$ does not divide c_i, $m+2 \leqq i \leqq n\}$. Let $J = \{j;\ j \notin I,\ m+1 \leqq j \leqq n\}$. Then $I = \{i;\ b_i$ is divided by the same power of p as $b_{i-1}\}$.

The important situation occurs when i belongs to J. For example, $m+1$ and $N(p)$ are the smallest and the greatest elements of J.

We will change the generators of $H^*(Y_{n,n-m}; \mathbf{Z}_p)$ to the following

$$c_i x_i = \theta v_i, \qquad i \in I,$$
$$x_i = \bar{x}_i, \qquad i \in J,\ i \neq N(p);$$

then we obtain:

$$H^*(Y_{n,n-m}; \mathbf{Z}_p) = \mathbf{Z}_p[\bar{y}]/[\bar{y}^{N(p)}] \otimes \bigwedge (x_{m+1}, \ldots, \hat{x}_{N(p)}, \ldots, x_n)$$

where

(7) $\quad \theta v_i = c_i x_i, \quad i \in I; \quad i^*_{\mathbf{Z}_p} x_i = \bar{z}_i, \quad m+1 \leqq i \leqq n, \quad i \neq N(p) \quad$ and $\quad \pi^*_{\mathbf{Z}_p}(\bar{\omega}) = \bar{y}.$

Again in diagram (C), this time with $A = \mathbf{Q}$, we have

$$\theta v_i = c_i\tilde{x}_i + \tilde{u}_i, \quad \tilde{u}_i \text{ belongs to } \operatorname{Ker} i^*_{\mathbf{Q}}, \qquad m+2 \leqq i \leqq n.$$

We define $c_i w_i = \theta v_i$, $m+2 \leqq i \leqq n$ and we obtain

(7′)
$$H^*(Y_{n,n-m}; \mathbf{Q}) = \mathbf{Q}[\tilde{y}]/[\tilde{y}^{m+1}] \otimes \bigwedge (w_{m+2}, \ldots, w_n)$$
$$\text{where } \theta v_i = c_i w_i, \qquad i^*_{\mathbf{Q}} w_i = \tilde{z}_i, \qquad \pi^*_{\mathbf{Q}}(\tilde{\omega}) = \tilde{y}.$$

3. Next, we compute the Bockstein spectral sequence of the couple

$$\begin{array}{ccc} H^*(Y_{n,n-m}) & \xrightarrow{(\cdot p)^*} & H^*(Y_{n,n-m}) \\ \delta \nwarrow & & \swarrow \theta \\ & H^*(Y_{n,n-m}; \mathbf{Z}_p) & \end{array}$$

It follows from (7) that

$$E_1 = H^*(Y_{n,n-m}; \mathbf{Z}_p) = \mathbf{Z}_p[y]/[y^N] \otimes \bigwedge (x_{m+1} \cdots \hat{x}_{N(p)} \cdots x_n)$$

and from (7′) that

$$E_\infty = H^*(Y_{n,n-m})/\text{Torsion} \otimes \mathbf{Z}_p = \mathbf{Z}_p[y]/[y^{m+1}] \otimes \bigwedge (w_{m+2}, \ldots, w_n).$$

Recall that the differentials are Bockstein homomorphisms β_r and an element $x \in E_1$, belongs to Im θ if and only if

$$\beta_r x = 0 \quad \text{for all } r. \tag{8}$$

An element $y \in H^*(Y_{n,n-m})$ has torsion p^r, that is $p^r ay = 0$ where p does not divide a, if and only if $\theta y \notin \operatorname{Im} \beta_j$ for $j < r$, but

$$\theta y \in \operatorname{Im} \beta_r. \tag{9}$$

First we will give some easy results:

If $x \in E_r$, call $\phi(x)$ its image in E_∞, then

$$\phi(\bar{y}) = \tilde{y}; \qquad \phi(x_i) = w_i, \qquad i \in I. \tag{10}$$

By (7) and (8)

$$\beta_r(\bar{y}) = 0, \qquad \beta_r(x_i) = 0, \qquad \text{all } r, \ i \in I. \tag{11}$$

By (10), since $w_i \neq 0$,

$$x_i \notin \operatorname{Im} \beta_r, \qquad \text{all } r, \ i \in I. \tag{12}$$

We arrange J so that $m+1 = i(0) < i(1) < \cdots < i(j) < \cdots < i(t) = N(p)$ and put $b_{i(j)} = p^{r(j)} a_j$, where p does not divide a_j; then $r(j) > r(j+1)$ and $b_i = p^{r(j)} a_i$, $i(j) \leqq i < i(j+1)$.

By (9) and Proposition 5:

$$y^i \notin \operatorname{Im} \beta_r, \qquad r < r(j), \qquad y^i \in \operatorname{Im} \beta_{r(j)}, \qquad i(j) \leqq i < i(j+1). \tag{13}$$

Trivially

$$E_1^q = E_\infty^q, \qquad q < 2i(0) - 1. \tag{14}$$

Now, we will compute β_r:

LEMMA 8. *The following formulae hold for every j*

$$\beta_r x_{i(j)} = 0, \qquad r < r(j), \tag{15}$$

$$\beta_{r(j)} x_{i(j)} = k_j y^{i(j)}, \qquad k_j \in \mathbf{Z}_p, \ k_j \neq 0, \tag{16}$$

$$E_1^q = E_{r(j)}^q, \qquad q < 2i(j+1) - 1. \tag{17}$$

Proof. By (13) there is an element x such that $\beta_{r(0)} x = y^{i(0)}$ but x can only be a scalar multiple of $x_{i(0)}$ then (15) and (16) hold for $j = 0$.

By the same argument (15) and (16) hold for $j = h$ provided that (17) holds for $j = h - 1$.

In turn, (15) for every $j \leqq h$ and (11) together imply (17) for $j = h$ because Bockstein homomorphisms are derivations.

COROLLARY 9. *For every j*

$$\beta_{r(j)}(x_{i(j)} y^s) = k_j y^{i(j)+s} \neq 0, \qquad 0 \leqq s < i(j+1) - i(j), \tag{18}$$

$$\beta_{r(j)}(x_{i(j)} y^{i(j+1)-i(j)}) = 0. \tag{19}$$

Proof. (18) follows from (16) and (17). (19) follows from (16).

We call $u_{i(j+1)}$ the image of $x_{i(j)}y^{i(j+1)-i(j)}$ in $E_{r(j)+1}$.

It remains to prove that $\beta_r=0$ unless $r=r(j)$ for some j. This is part of the following lemma.

LEMMA 10. *$\beta_r=0$ unless $r=r(j)$ and $E_\infty=E_r(o)$.*

We use induction. Assign $\bar{y}$ to $\bar{y}$ and w_i to x_i for $i\in I$, $i<i(1)$.

By (15), . . . , (19) and $\dim E_\infty \leqq E_{r(0)}$, this correspondence determines an isomorphism from E_∞ onto $E_{r(0)}$, up to degree $2i(1)-2$.

Moreover, $\beta_r=0$ up to degree $2i(1)-2$ unless $r=r(0)$.

Suppose we have elements $\bar{u}_{i(j)}$, $j=1,\ldots,h$, such that:

(i) $\operatorname{gr}\bar{u}_{i(j)}=2i(j)-1$.

(ii) $\bar{u}_{i(j)}x_i=-x_i\bar{u}_{i(j)}$; $\bar{u}_{i(j)}u_{i(j')}=-\bar{u}_{i(j')}\bar{u}_{i(j)}$, $j'<j$; $(\bar{u}_{i(j)})^2=0$.

(iii) If we assign $\bar{y}$ to $\bar{y}$; w_i to x_i for $i\in I$, $i<i(j+1)$ and $w_{i(j)}$ to $\bar{u}_{i(j)}$we determine an isomorphism from E_∞ onto $E_{r(0)}$ up to degree $2i(h+1)-2$.

Suppose besides that $\beta_r=0$ up to degree $2i(h+1)-2$ unless $r=r(j)$, $j=0,\ldots,t-1$.

From these assumptions and (15), . . . , (19) we have $\dim E^q_{r(0)}=\dim E^q_\infty$, $q\leqq 2i(h+1)-1$ and all differentials are determined on all elements of degree $\leqq 2i(h+1)-1$ except $u_i(h+1)$ belonging to $E_{r(h)+1}$ and its images in E_r, $r>r(h)+1$.

Thus, for every r, $\beta_r u_{i(h+1)}$ must lie in the subspace of E_r spanned by $\{\beta_r a\}$, where a ranges over products. That means $\beta_r u_i(h+1)=0$ for $r<r(h-1)$; and there is an element u' in $E^{2i(h+1)-1}_{r(h-1)}$, such that $\beta_{r(h-1)}u'=0$ and u' does not belong to the subalgebra generated by elements with degree $<2i(h+1)-1$. It is easy to see that u' satisfies (ii). Again, $\beta_r u'=0$ for $r<r(h-2)$ and we repeat the argument until we reach $E_{r(0)}$, then we obtain an element $\bar{u}_{i(h+1)}$ in $E_{r(0)}$ to which we may assign $w_{i(h+1)}$. Now we assign w_i to x_i for $i\in I$, $i<i(h+2)$ and obtain an isomorphism up to degree $2i(h+2)-1$. Then we have finished the proof of (i), (ii), (iii) with $j=h+1$. From (11) we see $\beta_r=0$ up to degree $2i(h+2)-1$, unless $r=r(j)$ for $j=0,\ldots,t-1$. This completes the proof.

We have identified $H^*(Y_{n,n-m};\mathrm{Q})$ with $E_{r(0)}$ as algebras, for every prime p. Then we have completed the proof of Theorem A.

REFERENCES

1. P. Baum and W. Browder, *The cohomology of quotients of classical groups*, Topology 3, (1965), 305–336.

2. A. Borel, *Sur la cohomologie des espaces fibres principaux et des espaces homogenes des groupes de Lie compacts*, Ann. of Math. 57 (1953), 115–207.

3. S. Gitler and D. Handel, *The projective Stiefel manifolds*. I, Topology 7 (1968), 39–46.

NORTHWESTERN UNIVERSITY,
EVANSTON, ILLINOIS
UNIVERSIDAD DE BUENOS AIRES,
BUENOS AIRES, ARGENTINA